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Minimization problems for (R, S)-symmetric and (R, S)-skew symmetric matrices

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Abstract

Let $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ be nontrivial involutions; thus $R = R^{-1} \neq \pm I$ and $S = S^{-1} \neq \pm I$. We say that $A \in \mathbb{C}^{m\times n}$ is (R, S)-symmetric ((R, S)-skew symmetric) if $RAS = A$ ($RAS = -A$). Let $\mathscr{S}$ be the class of $m \times n$ (R, S)-symmetric matrices or the class of $m \times n$ (R, S)-skew symmetric matrices. Let $Z \in \mathbb{C}^{n\times q}$ and $W \in \mathbb{C}^{m\times q}$. We study the following problems:

(i) Give necessary and sufficient conditions for the existence of an $A \in \mathscr{S}$ such that $AZ = W$, and find all such matrices if the conditions are met.

(ii) Find $\sigma(Z, W) = \min_{A\in\mathscr{S}} \|AZ - W\|$ and characterize the class $\mathscr{S}(Z, W) = \{A \in \mathscr{S} \mid \|AZ - W\| = \sigma(Z, W)\}$.

(iii) If $B \in \mathbb{C}^{m\times n}$ is arbitrary, find $\sigma(Z, W, B) = \min_{A\in\mathscr{S}(Z,W)} \|A - B\|$ and find $A \in \mathscr{S}(Z, W)$ such that $\|A - B\| = \sigma(Z, W, B)$.

We obtain explicit formulas for $\sigma(Z, W)$, $\sigma(B, Z, W)$, and all the matrices in question.

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1. Introduction

Throughout this paper $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are nontrivial involutions; thus $R = R^{-1} \neq \pm I$ and $S = S^{-1} \neq \pm I$. We say that $A \in \mathbb{C}^{m\times n}$ is (R, S)-symmetric

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((R, S)-skew symmetric) if $RAS = A$ ($RAS = -A$). This definition is motivated by a similar definition of Chen [4], who required in addition that $R = R^*$ and $S = S^*$. Chen discussed applications that give rise to these matrices and considered least squares problems involving them.

In the following $\mathscr{S}$ is either the class of $m \times n(R, S)$-symmetric matrices or the class of $m \times n$ (R, S)-skew symmetric matrices, and $\|\cdot\|$ is the Frobenius norm.

We consider the following problems.

Problem 1. If $Z \in \mathbb{C}^{n\times q}$ and $W \in \mathbb{C}^{m\times q}$, find

$$\sigma(Z, W) = \min_{A\in\mathscr{S}} \|AZ - W\|$$

and characterize the class

$$\mathscr{S}(Z, W) = \{A \in \mathscr{S} \,|\, \|AZ - W\| = \sigma(Z, W)\}.$$

Problem 2. If $B \in \mathbb{C}^{m\times n}$, $Z \in \mathbb{C}^{n\times q}$, and $W \in \mathbb{C}^{m\times q}$, find

$$\sigma(Z, W, B) = \min_{A\in\mathscr{S}(Z,W)} \|A - B\|$$

and find $A \in \mathscr{S}(Z, W)$ such that $\|A - B\| = \sigma(Z, W, B)$.

We obtain explicit formulas for $\sigma(Z, W)$, $\sigma(B, Z, W)$, all matrices in $\mathscr{S}(Z, W)$, and the solution of Problem 2. As a byproduct of our results on Problem 1 we obtain necessary and sufficient conditions on Z and W for existence of $A \in \mathscr{S}$ such that $AZ = W$, and an explicit formula for all such A. This last result requires no additional assumptions on R and S. However, we assume that $R = R^*$ in connection with Problem 1, and that $R = R^*$ and $S = S^*$ in connection with Problem 2.

Zhou et al. [8] have studied these problems for $n \times n$ centrosymmetric matrices, where $R = S = J$, the matrix with ones on the secondary diagonal and zeros elsewhere. Peng and Hu [6] have studied the existence of $n \times n$ (R, R)-symmetric ((R, R)-skew symmetric) matrices X such that $AX = B$, where A and B are given in $\mathbb{C}^{m\times n}$, and the approximation of an arbitrary $Y \in \mathbb{C}^{n\times n}$ by matrices X that satisfy $AX = B$; however, they did not consider the minimization of $\|AX - B\|$ when there is no exact solution X.

If $m = n$ and $R = S$, our results apply to R symmetric and R-skew symmetric matrices, which are discussed in [7]. In particular, if $R = S = J$, our results apply to centrosymmetric and centroskew matrices; see, e.g., [1–3] and their references.

2. Preliminary results

Since an involution is diagonalizable, there are positive integers r and s such that $r + s = m$ and matrices $P \in \mathbb{C}^{m\times r}$ and $Q \in \mathbb{C}^{m\times s}$ such that

$$P^*P = I, \quad Q^*Q = I, \tag{1}$$

$$RP = P, \quad \text{and} \quad RQ = -Q. \tag{2}$$

Thus, the columns of P (Q) form an orthonormal basis for the eigenspace of R associated with the eigenvalue $\lambda = 1$ ($\lambda = -1$). Although P and Q are not unique, suitable P and Q can be obtained by applying the Gram–Schmidt procedure to the columns of $I + R$ and $I - R$, respectively.

If

$$\widehat{P} = \frac{P^*(I+R)}{2} \quad \text{and} \quad \widehat{Q} = \frac{Q^*(I-R)}{2}$$

then

$$\widehat{P}P = I, \quad \widehat{P}Q = 0, \quad \widehat{Q}P = 0, \quad \text{and} \quad \widehat{Q}Q = I,$$

so

$$\begin{bmatrix} P & Q \end{bmatrix}^{-1} = \begin{bmatrix} \widehat{P} \\ \widehat{Q} \end{bmatrix}. \tag{3}$$

It is straightforward to verify that $R = R^*$ if and only if $P^*Q = 0$, and that in this case $\widehat{P} = P^*$ and $\widehat{Q} = Q^*$, so $\begin{bmatrix} P & Q \end{bmatrix}$ is unitary.

Similarly, there are positive integers k and ℓ such that $k + \ell = n$ and matrices $U \in \mathbb{C}^{n\times k}$ and $V \in \mathbb{C}^{n\times \ell}$ such that

$$U^*U = I, \quad V^*V = I,$$

$$SU = U, \quad \text{and} \quad SV = -V. \tag{4}$$

Moreover, if

$$\widehat{U} = \frac{U^*(I+S)}{2} \quad \text{and} \quad \widehat{V} = \frac{V^*(I-S)}{2}, \tag{5}$$

then

$$\widehat{U}U = I, \quad \widehat{U}V = 0, \quad \widehat{V}U = 0, \quad \text{and} \quad \widehat{V}V = I,$$

so

$$\begin{bmatrix} U & V \end{bmatrix}^{-1} = \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}. \tag{6}$$

Again, $S = S^*$ if and only if $U^*V = 0$, and in this case $\widehat{U} = U^*$ and $\widehat{V} = V^*$, so $\begin{bmatrix} U & V \end{bmatrix}$ is unitary.

From (3) and (6), any $m \times n$ matrix can be written as

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A_{PU} & A_{PV} \\ A_{QU} & A_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}. \tag{7}$$

From (4) and (5), $\widehat{U}S = \widehat{U}$ and $\widehat{V}S = -\widehat{V}$. This, (2), and (7) imply that

$$RAS = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A_{PU} & -A_{PV} \\ -A_{QU} & A_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}. \tag{8}$$

If $Z \in \mathbb{C}^{n\times q}$ then Z can be written uniquely as $Z = UX + VY$ with $X = \widehat{U}Z \in \mathbb{C}^{k\times q}$ and $Y = \widehat{V}Z \in \mathbb{C}^{\ell\times q}$. If $W \in \mathbb{C}^{m\times q}$ then W can be written uniquely as $W = P\Phi + Q\Psi$ with $\Phi = \widehat{P}W \in \mathbb{C}^{r\times q}$ and $\Psi = \widehat{Q}W \in \mathbb{C}^{s\times q}$.

Given $C \in \mathbb{C}^{p\times q}$, $C^\dagger$ denotes the Moore–Penrose inverse of C; thus, $C^\dagger \in \mathbb{C}^{q\times p}$ is the unique matrix such that

$$CC^\dagger C = C, \quad C^\dagger CC^\dagger = C^\dagger, \quad (CC^\dagger)^* = CC^\dagger, \quad \text{and} \quad (C^\dagger C)^* = C^\dagger C.$$

We use these relationships repeatedly without citation.

We need the following lemmas.

Lemma 1. *If $H \in \mathbb{C}^{p\times q}$ and $F \in \mathbb{C}^{q\times m}$ then $HF = 0$ if and only if $H = K\left(I - FF^\dagger\right)$ for some $K \in \mathbb{C}^{p\times q}$.*

Proof. If $H = K\left(I - FF^\dagger\right)$ then $HF = 0$. For the converse, suppose $HF = 0$. Choose K so that $Kv = Hv$ if $v^*F = 0$. (For example, $K = H$ is acceptable.) If $w \in \mathbb{C}^q$ then $w = v + Fc$ where $c \in \mathbb{C}^m$ and $v^*F = 0$. Then $Hw = Hv$ and

$$K(I - FF^\dagger)w = K(I - FF^\dagger)v = Kv - K(v^*FF^\dagger)^* = Kv = Hv = Hw.$$

Hence, $H = K(I - FF^\dagger)$. $\square$

Lemma 2. *If $F \in \mathbb{C}^{q\times m}$ and $G \in \mathbb{C}^{p\times m}$ then*

$$\min_{C\in\mathbb{C}^{p\times q}} \|CF - G\| = \|G(I - F^\dagger F)\|, \tag{9}$$

and this minimum is attained if and only if

$$C = GF^\dagger + K(I - FF^\dagger) \tag{10}$$

with $K \in \mathbb{C}^{p\times q}$ arbitrary. Moreover, $C_0 = GF^\dagger$ is the unique matrix of this form with minimum norm.

Proof. Write

$$CF - G = (C - GF^\dagger)F - G(I - F^\dagger F).$$

Since

$$(C - GF^\dagger)F(G(I - F^\dagger F))^* = 0,$$

it follows that

$$\|CF - G\|^2 = \|(C - GF^\dagger)F\|^2 + \|G(I - F^\dagger F)\|^2,$$

which implies (9) and that the minimum is attained if and only $(C - GF^\dagger)F = 0$. Now Lemma 1 implies (10). Since

$$GF^\dagger(K(I - FF^\dagger))^* = 0,$$

(10) implies that

$$\|C\|^2 = \|GF^\dagger\|^2 + \|K(I - FF^\dagger)\|^2 \geqslant \|GF^\dagger\|^2,$$

with equality if and only if $K(I - FF^\dagger) = 0$. □

Lemmas 1 and 2 imply the following result, which is related to [5, Lemma 1.3].

Lemma 3. *If $F \in \mathbb{C}^{q\times m}$ and $G \in \mathbb{C}^{p\times m}$, then $CF = G$ for some $C \in \mathbb{C}^{p\times q}$ if and only if $G(I - FF^\dagger) = 0$. In this case, C is given by* (10), *with $K \in \mathbb{C}^{p\times q}$ arbitrary.*

Lemma 4. *Suppose that $L \in \mathbb{C}^{p\times q}$ and $\Gamma \in \mathbb{C}^{q\times q}$ where $\Gamma^2 = \Gamma = \Gamma^*$. Then the unique matrix of the form $L - M\Gamma$ with minimum Frobenius norm is $L(I - \Gamma)$.*

Proof. Write

$$L - M\Gamma = L(I - \Gamma) + (L - M)\Gamma.$$

Since

$$L(I - \Gamma)((L - M)\Gamma)^* = 0,$$

it follows that

$$\|L - M\Gamma\|^2 = \|L(I - \Gamma)\|^2 + \|(L - M)\Gamma\|^2,$$

which implies the conclusion. □

Henceforth, if $G \in \mathbb{C}^{p\times q}$ we denote $\Gamma_G = (I - GG^\dagger)$. Note that $\Gamma_G^2 = \Gamma_G = \Gamma_G^*$.

3. (R, S)-symmetric matrices

Eqs. (1) and (6)–(8) yield the following characterization of (R, S)-symmetric matrices.

Theorem 1. *A is (R, S)-symmetric if and only if*

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A_{PU} & 0 \\ 0 & A_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} \tag{11}$$

where

$$A_{PU} = P^*AU \quad \text{and} \quad A_{QV} = Q^*AV.$$

Theorem 2. *Let $\mathscr{S} = \{A \in \mathbb{C}^{m\times n} | RAS = A\}$, $Z = UX + VY \in \mathbb{C}^{n\times q}$, and $W = P\Phi + Q\Psi \in \mathbb{C}^{m\times q}$. Then:*

(i) *There is an $A \in \mathscr{S}$ such that $AZ = W$ if and only if*

$$\Phi(I - X^\dagger X) = 0 \quad \text{and} \quad \Psi(I - Y^\dagger Y) = 0. \tag{12}$$

All such A are of the form

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} \Phi X^\dagger + K_{PU}\Gamma_X & 0 \\ 0 & \Psi Y^\dagger + K_{QV}\Gamma_Y \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}, \tag{13}$$

where $K_{PU} \in \mathbb{C}^{r\times k}$ *and* $K_{QV} \in \mathbb{C}^{s\times \ell}$ *are arbitrary.*

(ii) *If* $R = R^*$ *then*

$$\sigma(Z, W) = (\|\Phi(I - X^\dagger X)\|^2 + \|\Psi(I - Y^\dagger Y)\|^2)^{1/2}$$

and $\mathscr{S}(Z, W)$ *is the set of matrices of the form* (13).

(iii) *If* $R = R^*$ *and* $S = S^*$, *so that* (13) *becomes*

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} \Phi X^\dagger + K_{PU}\Gamma_X & 0 \\ 0 & \Psi Y^\dagger + K_{QV}\Gamma_Y \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix}, \tag{14}$$

then

$$A_0 = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} \Phi X^\dagger & 0 \\ 0 & \Psi Y^\dagger \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix}$$

is the unique member of $\mathscr{S}(Z, W)$ *with minimum norm.*

(iv) *If* $R = R^*$, $S = S^*$, *and*

$$B = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} B_{PU} & B_{PV} \\ B_{QU} & B_{QV} \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix} \tag{15}$$

is a given matrix in $\mathbb{C}^{m\times n}$, *then*

$$\begin{aligned} \sigma^2(B, Z, W) &= \|(B_{PU}X - \Phi)X^\dagger\|^2 + \|(B_{QV}Y - \Psi)Y^\dagger\|^2 \\ &\quad + \|B_{PV}\|^2 + \|B_{QU}\|^2, \end{aligned} \tag{16}$$

and this minimum is attained if and only if

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} \Phi X^\dagger + B_{PU}\Gamma_X & 0 \\ 0 & \Psi Y^\dagger + B_{QV}\Gamma_Y \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix}.$$

Proof

(i) With A as in (11),

$$AZ - W = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A_{PU}X - \Phi \\ A_{QV}Y - \Psi \end{bmatrix}. \tag{17}$$

Therefore $AZ = W$ if and only if $A_{PU}X = \Phi$ and $A_{QV}Y = \Psi$. Now Lemma 3 implies the conclusion.

(ii) If $R = R^*$ then $\begin{bmatrix} P & Q \end{bmatrix}$ is unitary and (17) implies that

$$\|AZ - W\|^2 = \|A_{PU}X - \Phi\|^2 + \|A_{QV}Y - \Psi\|^2,$$

so A_{PU} and A_{QV} must be chosen to minimize the two terms on the right independently. Applying Lemma 2 to each of these minimization problems yields the conclusion.

(iii) Since $R = R^*$ and $S = S^*$, $\begin{bmatrix} P & Q \end{bmatrix}$ and $\begin{bmatrix} U & V \end{bmatrix}$ are unitary. Hence, from (14),

$$\|A\|^2 = \|\Phi X^\dagger + K_{PU}\Gamma_X\|^2 + \|\Psi Y^\dagger + K_{QV}\Gamma_Y\|^2.$$

Since

$$(\Phi X^\dagger)(K_{PU}\Gamma_X)^* = 0 \quad \text{and} \quad (\Psi Y^\dagger)(K_{QV}\Gamma_Y)^* = 0,$$

it follows that

$$\|A\|^2 = \|A_0\|^2 + \|K_{PU}\Gamma_X\|^2 + \|K_{QV}\Gamma_Y\|^2,$$

which implies the conclusion.

(iv) From (14) and (15),

$$B - A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} B_{PU} - \Phi X^\dagger - K_{PU}\Gamma_X & B_{PV} \\ B_{QU} & B_{QV} - \Psi Y^\dagger - K_{QV}\Gamma_Y \end{bmatrix} \times \begin{bmatrix} U^* \\ V^* \end{bmatrix}.$$

Therefore

$$\|B - A\|^2 = \|B_{PU} - \Phi X^\dagger - K_{PU}\Gamma_X\|^2 + \|B_{QV} - \Psi Y^\dagger - K_{QV}\Gamma_Y\|^2 + \|B_{PV}\|^2 + \|B_{QU}\|^2,$$

so K_{PU} and K_{QV} must be chosen to minimize the first two terms on the right independently. Applying Lemma 4 to each of these minimization problems yields (16) and shows that the minimum is attained if and only if

$$K_{PU}\Gamma_X = (B_{PU} - \Phi X^\dagger)\Gamma_X = B_{PU}\Gamma_X$$

and

$$K_{QV}\Gamma_Y = (B_{QV} - \Psi Y^\dagger)\Gamma_Y = B_{QV}\Gamma_Y. \qquad \square$$

Corollary 1. *In the context of Theorem* 2, *suppose that* $\operatorname{rank}(X) = \operatorname{rank}(Y) = q$. *Then* $\sigma(Z, W) = 0$, $AZ = W$ *with* $A \in \mathscr{S}$ *if and only if* A *is of the form* (13), *and Theorem* 2(iii)–(iv) *hold*.

Proof. If $\operatorname{rank}(X) = \operatorname{rank}(Y) = q$ then $X^\dagger X = Y^\dagger Y = I$, so (12) holds for any Φ and Ψ. $\square$

4. (R, S)-skew symmetric matrices

Eqs. (1) and (6)–(8) yield the following characterization of (R, S)-skew symmetric matrices.

Theorem 3. *A is (R, S)-skew symmetric if and only if*

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} 0 & A_{PV} \\ A_{QU} & 0 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix},$$

where

$$A_{PV} = P^*AV \quad \text{and} \quad A_{QU} = Q^*AU.$$

The proof of the following theorem is similar to the proof of Theorem 2.

Theorem 4. *Let $\mathscr{S} = \{A \in \mathbb{C}^{m\times n} | RAS = -A\}$, $Z = UX + VY \in \mathbb{C}^{n\times q}$, and $W = P\Phi + Q\Psi \in \mathbb{C}^{m\times q}$. Then:*

(i) *There is an $A \in \mathscr{S}$ such that $AZ = W$ if and only if*

$$\Phi(I - Y^\dagger Y) = 0 \quad \text{and} \quad \Psi(I - X^\dagger X) = 0.$$

All such matrices are of the form

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} 0 & \Phi Y^\dagger + K_{PV}\Gamma_Y \\ \Psi X^\dagger + K_{QU}\Gamma_X & 0 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}, \tag{18}$$

where $K_{PV} \in \mathbb{C}^{r\times \ell}$ and $K_{QU} \in \mathbb{C}^{s\times k}$ are arbitrary.

(ii) *If $R = R^*$ then*

$$\sigma(Z, W) = (\|\Phi(I - Y^\dagger Y)\|^2 + \|\Psi(I - X^\dagger X\|^2)^{1/2},$$

and $\mathscr{S}(Z, W)$ is the set of matrices of the form (18).

(iii) *If $R = R^*$ and $S = S^*$, so that* (18) *becomes*

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} 0 & \Phi Y^\dagger + K_{PV}\Gamma_Y \\ \Psi X^\dagger + K_{QU}\Gamma_X & 0 \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix},$$

then

$$A_0 = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} 0 & \Phi Y^\dagger \\ \Psi X^\dagger & 0 \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix}$$

is the unique member of $\mathscr{S}(Z, W)$ with minimum norm.

(iv) *If $R = R^*$, $S = S^*$, and* (15) *is a given matrix in $\mathbb{C}^{m\times n}$, then*

$$\begin{aligned}\sigma^2(B, Z, W) = {} & \|(B_{PV}Y - \Phi)Y^\dagger\|^2 + \|(B_{QU}X - \Psi)X^\dagger\|^2 \\ & + \|B_{PU}\|^2 + \|B_{QV}\|^2\end{aligned}$$

and this minimum is attained if and only if

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} 0 & \Phi Y^\dagger + B_{PV}\Gamma_Y \\ \Psi X^\dagger + B_{QU}\Gamma_X & 0 \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix}.$$

Corollary 2. *In the context of Theorem* 4, *suppose that* $\operatorname{rank}(X) = \operatorname{rank}(Y) = q$. *Then $\sigma(Z, W) = 0$, $AZ = W$ with $A \in \mathscr{S}$ if and only if A is of the form* (18), *and Theorem* 4(iii)–(iv) *hold.*

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References

[1] A.L. Andrew, Eigenvectors of certain matrices, Linear Algebra Appl. 7 (1973) 151–162.
[2] A.L. Andrew, Solution of equations involving centrosymmetric matrices, Technometrics 15 (1973) 405–407.
[3] A.L. Andrew, Centrosymmetric matrices, SIAM Rev. 40 (1998) 697–698.
[4] H.-C. Chen, Generalized reflexive matrices: special properties and applications, SIAM J. Matrix Anal. Appl. 19 (1998) 140–153.
[5] J.-G. Sun, Backward perturbation analysis of certain characteristic subspaces, Numer. Math. 65 (1993) 357–382.
[6] Z.-Y. Peng, X.-Y. Hu, The reflexive and anti-reflexive solutions of the matrix equation $AX = B$, Linear Algebra Appl. 375 (2003) 147–155.
[7] W.F. Trench, Characterization and properties of matrices with generalized symmetry or skew symmetry, Linear Algebra Appl. 377 (2004) 207–218.
[8] F.-Z. Zhou, L. Zhang, X.-Y. Hu, Least-square solutions for inverse problems of centrosymmetric matrices, Comput. Math. Appl. 45 (2003) 1581–1589.